① $y'' - xy = 0, \; y(0) = 2, \; y'(0) = 1$

we want a power series centered at $x_0 = 0$

Step 1: First what will the radius of our power series solution be?

$$y'' - \underbrace{x}_{\text{coefficients}} \cdot y = \underbrace{0}$$

$$x = 0 + 1 \cdot x$$
$$0 = 0$$

x and 0 are polynomials so they have finite power series centered at $x_0 = 0$ with radius of convergence $r = \infty$

Thus our solution will have radius of convergence $r = \infty$

Step 2: Find the power series solution centered at $x_0 = 0$:

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$

We are given $\boxed{y(0) = 2,\ y'(0) = 1}$

To find $y''(0)$ we use $y'' - xy = 0$

So, $y'' = xy$.

So, $y''(0) = 0 \cdot \underbrace{[y(0)]}_{2} = 0 \cdot 2 = 0$

$\boxed{y''(0) = 0}$

Differentiate $y'' = xy$

to get $y''' = y + xy'$

So, $y'''(0) = \underbrace{y(0)}_{2} + (0)\underbrace{[y'(0)]}_{1} = 2 + 0 = 2$

$\boxed{y'''(0) = 2}$

Differentiate $y''' = y + xy'$ to get

$y'''' = y' + y' + xy''$

So, $y'''' = 2y' + xy''$

Thus, $y''''(0) = 2\underbrace{[y'(0)]}_{1} + (0)\underbrace{[y''(0)]}_{0} = 2$

$\boxed{y''''(0) = 2}$

Differentiate $y'''' = 2y' + xy''$

to get $y''''' = 2y'' + y'' + xy'''$

So, $y''''' = 3y'' + xy'''$

So, $y'''''(0) = 3\Big[\underbrace{y''(0)}_{0}\Big] + (0)\Big[\underbrace{y'''(0)}_{2}\Big] = 0$

$y'''''(0) = 0$

Differentiate $y''''' = 3y'' + xy'''$

to get $y'''''' = 3y''' + y''' + xy''''$

$y'''''' = 4y''' + xy''''$

$y''''''(0) = 4\underbrace{y'''(0)}_{2} + (0)\Big[\underbrace{y''''(0)}_{2}\Big] = 8$

$y''''''(0) = 8$

Now let's fill in what we have so far:

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y''''(0)}{4!}x^4 + \frac{y'''''(0)}{5!}x^5 + \frac{y''''''(0)}{6!}x^6 + \dots$$

$$y(x) = 2 + 1 \cdot x + 0x^2 + \frac{2}{3!}x^3 + \frac{2}{4!}x^4 + \frac{0}{5!}x^5 + \frac{8}{6!}x^6 + \cdots$$

$$\boxed{\begin{gathered} y(x) = 2 + x + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{90}x^6 + \\ \text{with radius of convergence } r = \infty. \end{gathered}}$$

(2) We want a power series solution for

$$\left.\begin{array}{l} y'' - (x+1)y' + x^2 y = 0 \\ y'(0) = 1, \quad y(0) = 1 \end{array}\right]$$

Here $x_0 = 0$

Step 1: Find the radius of convergence of the solution.

$$y'' \underbrace{-(x+1)}\, y' + \underbrace{x^2}\, y = \underbrace{0}$$

coefficents

$$\left.\begin{array}{l} -(x+1) = -1 + x \\ x^2 = 0 + 0x + 1 \cdot x^2 \\ 0 = 0 \end{array}\right]$$

these are all polynomials so they have finite power series and radius of convergence $r = \infty$

So our solution will have radius of convergence $r = \infty$.

Step 2: Find the solution y where

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$

We have

$$\boxed{\begin{array}{l} \left.\begin{array}{l} y'' = (x+1)y' - x^2 y \\ y(0) = 1 \end{array}\right] \text{given} \\ y'(0) = 1 \\ y''(0) = (0+1)\underbrace{y'(0)}_{1} - 0^2 \underbrace{y(0)}_{1} = 1 \end{array}}$$

$$\boxed{\begin{array}{l} y(0) = 1 \\ y'(0) = 1 \\ y''(0) = 1 \end{array}}$$

Differentiate $y''=(x+1)y'-x^2y$ to get the next step

$$y''' = y' + (x+1)y'' - 2xy - x^2y'$$

$$= (x+1)y'' + (1-x^2)y' - 2xy$$

$$y'''(0) = (0+1)\underbrace{y''(0)}_{1} + (1-0^2)\underbrace{y'(0)}_{1} - 2(0)\underbrace{y(0)}_{1}$$

$$= 2$$

$$\boxed{y'''(0) = 2}$$

Differentiate the y''' formula above to get:

$$y'''' = y'' + (x+1)y''' + (-2x)y' + (1-x^2)y''$$
$$- 2y - 2xy'$$

$$y''''(0) = \underbrace{y''(0)}_{1} + (0+1)\underbrace{y'''(0)}_{2} - 2(0)\underbrace{y'(0)}_{1} + (1-0^2)\underbrace{y''(0)}_{1}$$
$$- 2\underbrace{y(0)}_{1} - 2(0)\underbrace{y'(0)}_{1}$$

$$= 1 + 2 - 0 + 1 - 2 - 0$$

$$= 2$$

$$\boxed{y''''(0) = 2}$$

Thus,

$$y(x) = 1 + x + \frac{1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{2}{4!}x^4 + \dots$$

$$y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \dots$$

with radius of convergence $r = \infty$, so it converges for $-\infty < x < \infty$.

③ $y'' + \sin(x) y' + e^x y = 0$

$y'(0) = 1, \ y(0) = 1$

Here $x_0 = 0$

Step 1: Find the radius of convergence of the solution.

$$y'' + \underbrace{\sin(x)}\, y' + \underbrace{e^x}\, y = \underbrace{0}$$

coefficients

$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$

$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$

0

these all have $r = \infty$

Thus, the solution will have radius of convergence $r = \infty$.

Step 2: Find the solution y where

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y''''(0)}{4!}x^4 + \dots$$

We have:

$$y'' + \sin(x)y' + e^x y = 0$$

$$y'(0) = 1, \quad y(0) = 1$$

$$y''(0) + \underbrace{\sin(0)}_{0}\underbrace{y'(0)}_{1} + \underbrace{e^0}_{1}\underbrace{y(0)}_{1} = 0$$

$$y''(0) = -1$$

$$\boxed{\begin{array}{l} y''(0) \\ = -1 \end{array}}$$

Differentiate $y'' + \sin(x)y' + e^x y = 0$ to find y'''. We get

$$y''' + \cos(x)y' + \sin(x)y'' + e^x y + e^x y' = 0$$

$$y''' + \sin(x)y'' + (\cos(x) + e^x)y' + e^x y = 0$$

$$y'''(0) + \sin(0)y''(0) + (\cos(0) + e^0)y'(0) + e^0 y(0) = 0$$

$$y'''(0) + 0 \cdot (-1) + (1+1)(1) + (1)(1) = 0$$

$$y'''(0) = -3$$

$$\boxed{y'''(0) = -3}$$

So,

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$

$$y(x) = 1 + x + \frac{-1}{2!}x^2 - \frac{3}{3!}x^3 + \dots$$

$$\boxed{y(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \dots}$$

with radius of convergence $r = \infty$

(4) We want a power series solution to the initial-value problem

$$xy'' + x^2 y' - 2y = 0$$
$$y'(1) = 1, \quad y(1) = 1$$

Here $x_0 = 1$

Divide by x to get

$$y'' + \underbrace{x}\, y' \underbrace{- \frac{2}{x}}\, y = \underbrace{0}$$

coefficents

$x = 0 + 1 \cdot x$ — $r = \infty$, polynomial

$-\frac{2}{x} = -2\left[1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots\right]$

$= 2 - 2(x-1) + 2(x-1)^2 - 2(x-1)^3 + \cdots$

$r = 1$ from class/HW

$0 = 0$ — $r = \infty$, polynomial

The minimum of $r = \infty, r = 1, r = \infty$ is $r = 1$.
Thus, the solution will have at least radius of convergence $r = 1$.

<u>Step 2</u>: Find the solution y where

$$y(x) = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y'''(1)}{3!}(x-1)^3 + \cdots$$

We have:

$$y'' + x y' - 2x^{-1} y = 0$$

$$y'(1) = 1, \; y(1) = 1$$

$$y''(1) + 1 \cdot \underbrace{y'(1)}_{1} - 2(1)^{-1} \underbrace{y(1)}_{1} = 0$$

$$y''(1) + 1 - 2 = 0$$

$$y''(1) = 1$$

$y(1) = 1$
$y'(1) = 1$
$y''(1) = 1$

Differentiate $y'' + x y' - 2x^{-1} y = 0$ to get the next step.

$$y''' + y' + x y'' + 2x^{-2} y - 2x^{-1} y' = 0$$

$$y''' + x y'' + (1 - 2x^{-1}) y' + 2x^{-2} y = 0$$

$$y'''(1) + 1 \cdot \underbrace{y''(1)}_{1} + (1 - 2(1)^{-1}) \underbrace{y'(1)}_{1} + 2(1)^{-2} \underbrace{y(1)}_{1} = 0$$

$$y'''(1) + 1 - 1 + 2 = 0 \longrightarrow y'''(1) = -2$$

$y'''(1) = -2$

Differentiate the y''' formula above to find a formula for y''''.

We get

$$y'''' + y'' + xy''' + (2x^{-2})y' + (1-2x^{-1})y'' - 4x^{-3}y + 2x^{-2}y' = 0$$

$$y'''' + xy''' + (2-2x^{-1})y'' + 4x^{-2}y' - 4x^{-3}y = 0$$

$$y''''(1) + (1)\underbrace{y'''(1)}_{-2} + (2-2(1)^{-1})\underbrace{y''(1)}_{1} + 4(1)^{-2}\underbrace{y'(1)}_{1} - 4(1)^{-3}\underbrace{y(1)}_{1} = 0$$

$$y''''(1) - 2 + 0 + 4 - 4 = 0$$

$$y''''(1) = 2$$

$$\boxed{y''''(1) = 2}$$

Thus, for $-1 < x < 1$ we have

$$y(x) = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y'''(1)}{3!}(x-1)^3 + \frac{y''''(1)}{4!}(x-1)^4 + \cdots$$

$$= 1 + (x-1) + \frac{1}{2!}(x-1)^2 + \frac{-2}{3!}(x-1)^3 + \frac{2}{4!}(x-1)^4 + \dots$$

$$= 1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{-1}{3}(x-1)^3 + \frac{1}{12}(x-1)^4 + \dots$$

$3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

So,

$$y(x) = 1 + (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{3}(x-1)^3 + \frac{1}{12}(x-1)^4 + \dots$$

with radius of convergence at least $r = 1$